

Perturbative quantum corrections to the supersymmetric CP^1 kink with twisted mass

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ABSTRACT: We present an explicit calculation of the one-loop quantum corrections to the mass and the two central charges of the kink solution of an $\mathcal{N} = (2, 2)$ supersymmetric CP^1 model with twisted mass, using supersymmetry preserving dimensional regularization adapted to solitons. We find that the quantum corrections of the mass and one of the central charges are nontrivial (but saturate the BPS bound), while the other central charge receives no corrections. The nontrivial central charge correction corresponds to a quantum anomaly, which in our scheme appears as parity violation in the regulating extra dimension, and its magnitude is in agreement with exact results obtained by Dorey on the basis of a massive analog of mirror symmetry from a dual $U(1)$ gauge theory, confirming also recent work by Shifman, Vainshtein, and Zwicky.

KEYWORDS: Solitons Monopoles and Instantons, Field Theories in Lower Dimensions, Supersymmetry and Duality.

Dedicated to the memory of Wolfgang Kummer

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1. Introduction

Nonperturbative results on supersymmetric (susy) theories such as the famous Seiberg-Witten solution of $\mathcal{N} = 2$ super-Yang-Mills theories in 4 dimensions [1, 2] rely crucially on the presence of solitons which saturate the Bogomolnyi bound at both the classical level [3] and at the quantum level [4], namely when equality of mass and central charge gives rise to multiplet shortening. Results which are remarkably similar to those of Seiberg and Witten have been obtained by Dorey [5] for a two-dimensional $\mathcal{N} = (2, 2)$ U(1) gauge theory with N chiral multiplets of equal charge and twisted mass terms [6]. At large gauge coupling e this theory is in a Higgs phase whose low-energy limit is described by a classically massive $\mathcal{N} = (2, 2)$ CP^{N-1} model with BPS-saturated dyons which can carry both topological and Noether charges. The dual (mirror) theory, where the (dimensionful) gauge coupling e is smaller than all other mass scales, can be solved exactly and because the BPS spectrum is independent of e this yields all-order results for the spectrum of the CP^{N-1} model as a function of the twisted masses. Moreover these results turned out to be described by the same elliptic curve that appeared in the Seiberg-Witten solution of $\mathcal{N} = 2$ gauge theories in four dimensions.

Recently, there has been renewed interest in this model since it arises also as the effective field theory of so-called confined nonabelian monopoles, which reside within non-abelian flux tubes (vortices) of $\mathcal{N} = 2$ gauge theories with gauge group $SU(N) \times U(1)$ and

N flavors [7–13]. This connection in fact explained the observation of ref. [5] of a striking parallel between four-dimensional $\mathcal{N} = 2$ super-Yang-Mills theory and the two-dimensional $\mathcal{N} = (2, 2)$ CP^{N-1} model, because the four-dimensional Fayet-Iliopoulos parameter does not enter the formulae for the spectrum of the BPS sector so that they cover both the Higgs and the Coulomb branches. The theories giving rise to confined monopoles in the Higgs phase have an analytically accessible quasiclassical regime which corresponds to twisted masses that are much larger than the scale of the asymptotically free CP^{N-1} model. There the coupling constant of this effective theory is small and permits perturbative calculations.

A perturbative calculation of the quantum mass of the kink solution of the $\mathcal{N} = (2, 2)$ CP^1 model with twisted mass and a comparison with the exact results obtained from the dual theory has been made already in the original paper by Dorey [5], however without attempting accuracy beyond the logarithmic term that shows up at one-loop order. As has been pointed out recently by Shifman, Vainshtein and Zwicky [14], the finite contribution that remains after absorbing the logarithmic term into the renormalized coupling is associated with an anomalous contribution to the central charge analogous to the one found some time ago in ordinary susy kinks [15–17] and which was subsequently located also in $\mathcal{N} = 2$ super-Yang-Mills theories both in its Coulomb phase [18] and its Higgs phase [11].

In the present paper we complete the analysis begun by Dorey [5], namely a direct calculation of the quantum mass of the CP^1 kink with twisted mass and also of the central charges. Such a calculation involves the fluctuations of fermionic and bosonic fields in the background of the kink which despite isospectrality do not cancel due to a nonvanishing difference of the spectral densities. The resulting expression is in fact ultraviolet divergent and already in the minimally susy kink model presents a number of intricacies and pitfalls. For example, a sharp energy cutoff regularization incorrectly produces a null result for the finite terms of the one-loop contribution to the mass [19, 20] (and would do so also in the case of the susy CP^1 kink). The inconsistency of this method and its result with known results from the (nonsupersymmetric) sine-Gordon model was pointed out in ref. [21], which in 1997 reopened the issue of how to calculate quantum corrections for susy solitons. However, the alternative calculation presented in ref. [21] which used mode number regularization in finite volumes was polluted by boundary energy that occurs with periodic or antiperiodic boundary conditions. In ref. [22] this issue was resolved (by use of topological boundary conditions) which showed that the net quantum correction to the mass of a minimally susy kink is negative. Since there appeared to be no quantum correction to the central charge [20], this presented a problem with the BPS bound, which the authors of ref. [22] conjectured to be the result of a quantum anomaly. The latter was finally located by Shifman, Vainshtein, and Voloshin [15] as an anomalous additive contribution to the central charge operator which restores BPS saturation (which did not seem to be required by standard multiplet shortening arguments [4], but could eventually be explained through the possibility of single-state supermultiplets [23, 24]). These anomalous contributions to the central charge were confirmed in later works, e.g. ref. [16], although by using dimensional regularization methods ref. [25] seemed to obtain the required finite corrections to both mass and central charge without the need of an anomalous contribution.

In ref. [26, 17, 18], three of us performed one-loop calculations using a variant of

dimensional regularization in the presence of solitons which embeds the solitons in higher dimensions, from where susy-preserving dimensional reduction is possible. This reproduces the correct results for the quantum mass while indeed giving null results for the original central charge operator. However, anomalous contributions arise from nonvanishing bulk contributions to the momentum density in the extra dimension which break reflection invariance in the extra dimension, related to the fact that fermionic zero modes turn into chiral domain wall fermions. (Some additional issues arise for susy vortices in 2+1 dimensions and the $\mathcal{N} = 4$ monopole in 3+1 dimensions, see refs. [27–29]). In the present paper we apply our scheme to the susy CP^1 model with twisted mass term.

In superspace, the massless $\mathcal{N} = 1$ CP^1 model in 4 dimensions or the $\mathcal{N} = (2, 2)$ model in 2 dimensions can be written as

$$\mathcal{L} = \int d^4\theta K(\Phi, \bar{\Phi}), \quad K = r \ln(1 + \bar{\Phi}\Phi) \tag{1.1}$$

with Φ a conventional chiral superfield, $\bar{D}_\alpha\Phi = 0$.

In components, this reads, using the conventions of [30],

$$\mathcal{L} = -\frac{r}{\rho^2} \left\{ \partial_m \bar{\phi} \partial^m \phi + i \bar{\psi}_\alpha \bar{\sigma}^{m\dot{\alpha}\alpha} \left(\partial_m - \frac{2}{\rho} \bar{\phi}_\alpha (\partial_m \phi) \right) \psi + \frac{1}{2\rho^2} \psi \psi \bar{\psi} \bar{\psi} \right\}, \quad \rho \equiv 1 + \phi^\dagger \phi, \tag{1.2}$$

where $m = 0, \dots, 3$, and two of the ∂_m put to zero in the dimensional reduction to 2 dimensions. In 2 dimensions, the gauge coupling g defined by $r = \frac{2}{g^2}$ is dimensionless and its beta function is negative, so that the model is asymptotically free. Correspondingly, at the quantum level this theory has a mass gap determined by the renormalization group invariant scale Λ .

A classically massive version of the model in dimensions lower than 4 which preserves the entire supersymmetry can be obtained by introducing a background gauge field with nonvanishing value in the components corresponding to the dimensions eliminated in the reduction process,

$$\partial_m \rightarrow \partial_m + i \hat{V}_m, \quad \hat{V}_m \partial^m \Phi \equiv 0. \tag{1.3}$$

The mass terms provided by $\hat{V}_m = \text{const.} \neq 0$ have been termed twisted [6], because a gauge field strength superfield Σ in two dimensions is a twisted chiral superfield [31], satisfying $\bar{D}_R \Sigma = D_L \Sigma = 0$ instead of the conventional chiral constraint.

Dimensional reduction from 4 to 2 dimensions thus gives the possibility for introducing two mass parameters, which can be combined into one complex mass parameter $\tilde{m} = |m|e^{i\beta}$. The phase β corresponds to possible rotations in the two dimensions used for the dimensional reduction, and it turns out that because of the anomalous nature of the corresponding $U(1)_A$ transformation its effect can be absorbed into a θ term that can be added to the 2-dimensional Lagrangian.

The introduction of a mass term has the effect of providing the (nonnegative) potential term

$$V = \frac{r}{\rho^2} |m|^2 \phi^\dagger \phi = \frac{r|m|^2 \phi^\dagger \phi}{(1 + \phi^\dagger \phi)^2} \tag{1.4}$$

with zeros at $\phi = 0$ and $\phi = \infty$, which correspond to the north and south pole of the Riemann sphere, or CP^1 , obtained by compactifying the complex plane parametrized by ϕ .

The CP^1 kink (a close relative of the Q kinks studied originally in ref. [32]) is the static field configuration which asymptotes to these two different minima for left and right infinity. We shall study its one-loop quantum corrections in the perturbative regime provided by $m \gg \Lambda$, whereby the coupling g remains small for all energies.

2. The model in 3 dimensions

Dimensional reduction of the $\mathcal{N} = (1, 1)$ model (1.2) in 4 dimensions with the modification (1.3) leads to the $\mathcal{N} = (2, 2)$ sigma model with twisted mass term and the CP^1 kink solution in 2 dimensions, but in the following we shall reduce only from 4 to 3 dimensions, keeping the extra dimension for the purpose of susy preserving dimensional regularization by dimensional reduction. The dimension needed to generate the twisted mass term as a vev of a (background) gauge field component is thus compactified to vanishing size, but the other extra dimension is kept. The CP^1 kink of the 1+1-dimensional model becomes a CP^1 domain wall (a line) in 2+1 dimensions.

The action of the 2+1-dimensional model contains one complex scalar and one complex 2-component spinor¹

$$\mathcal{L} = -\frac{r}{\rho^2} \left[\partial_\mu \phi^\dagger \partial^\mu \phi + m^2 \phi^\dagger \phi + \bar{\psi} \gamma^\mu \partial_\mu \psi + m \bar{\psi} \psi \left(1 - \frac{2\phi^\dagger \phi}{\rho} \right) - \frac{2}{\rho} (\bar{\psi} \gamma^\mu \psi) (\phi^\dagger \partial_\mu \phi) - \frac{1}{\rho^2} (\bar{\psi} \psi) (\bar{\psi} \psi) \right], \quad \mu = 0, 1, 2, \quad \rho \equiv 1 + \phi^\dagger \phi, \quad (2.1)$$

where we have arranged for standard kinetic and mass terms by choosing a slightly unconventional ordering of Pauli matrices for $\bar{\sigma}^M = (-\mathbf{1}, -\sigma^1, -\sigma^3, -\sigma^2)$ in (1.2) together with $\gamma^0 = -i\sigma^2$. This fixes our conventions for the γ matrices in (2.1) as

$$\gamma^0 = -i\sigma_2, \quad \gamma^1 = -\sigma_3, \quad \gamma^2 = \sigma_1, \quad (2.2)$$

in agreement with the conventions used in our previous papers on susy kinks and their embedding in 2+1 dimensions [26, 17, 33] except for the overall sign of γ^1 . The direction of $x^2 \equiv y$ will be our regulator dimension, and the two-dimensional kink to be introduced shortly will depend only on $x^1 \equiv x$. The reason for using σ_3 in γ^1 rather than σ_1 is that this simplifies the fermionic fluctuation equations in the kink background (see below). Note that in our conventions the spinor components $\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$ correspond to positive and negative two-dimensional chirality with respect to the regulating dimension x^2 (moving “up” and “down” the domain wall); the more conventional left and right moving components of the final two-dimensional theory are related to the former by $\psi^R = (\psi^+ + \psi^-)/\sqrt{2}$ and $\psi^L = (\psi^+ - \psi^-)/\sqrt{2}$.

The Lagrangian density (2.1) is hermitian up to the antihermitian surface term $\partial_\mu \left(\frac{r}{\rho^2} \bar{\psi} \gamma^\mu \psi \right)$. One can write this model in a ψ - $\bar{\psi}$ symmetric way, or with the deriva-

¹Our conventions are $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta^{\mu\nu} = \text{diag}(-1, +1, +1)$, $\bar{\psi} = \psi^\dagger i\gamma^0$, thus $(\gamma^0)^2 = -1$ and $\gamma^{\mu\nu\rho} = -\epsilon^{\mu\nu\rho}$, $\gamma^{\mu\rho} = -\epsilon^{\mu\rho\sigma} \gamma_\sigma$ with $\epsilon^{012} = +1$.

tives acting on $\bar{\psi}$ instead of ψ , the only modifications being then, respectively,

$$-\frac{r}{\rho^2} \left[\dots + \frac{1}{2} \left(\bar{\psi} \gamma^\mu \overleftrightarrow{\partial}_\mu \psi \right) \dots - \frac{1}{\rho} (\bar{\psi} \gamma^\mu \psi) (\phi^\dagger \overleftrightarrow{\partial}_\mu \phi) \dots \right] \quad (2.3)$$

and

$$-\frac{r}{\rho^2} \left[\dots - \left(\bar{\psi} \gamma^\mu \overleftarrow{\partial}_\mu \psi \right) \dots + \frac{2}{\rho} (\bar{\psi} \gamma^\mu \psi) (\phi^\dagger \overleftarrow{\partial}_\mu \phi) \dots \right], \quad (2.4)$$

where it is understood that derivatives never act outside parentheses.

These actions are invariant under the following $\mathcal{N} = (2, 2)$ rigid susy transformations with two complex parameters ϵ^+ , ϵ^- with $\epsilon = \begin{pmatrix} \epsilon^+ \\ \epsilon^- \end{pmatrix}$,

$$\begin{aligned} \delta\phi &= \bar{\epsilon}\psi, & \delta\phi^\dagger &= \bar{\psi}\epsilon, \\ \delta\psi &= \gamma^\mu \partial_\mu \phi \epsilon - m\phi\epsilon + \frac{2\phi^\dagger}{\rho} (\bar{\epsilon}\psi)\psi, \\ \delta\bar{\psi} &= -\bar{\epsilon}\gamma^\mu \partial_\mu \phi^\dagger - \bar{\epsilon}\phi^\dagger m + \frac{2\phi}{\rho} (\bar{\psi}\epsilon)\bar{\psi}. \end{aligned} \quad (2.5)$$

3. The susy algebra

The susy algebra on ϕ, ϕ^\dagger, ψ has the following form

$$\begin{aligned} [\delta(\bar{\epsilon}_1), \delta(\bar{\epsilon}_2)] &= [\delta(\epsilon_1), \delta(\epsilon_2)] = 0, \\ [\delta(\epsilon_1), \delta(\bar{\epsilon}_2)] \begin{pmatrix} \phi \\ \phi^\dagger \end{pmatrix} &= (\bar{\epsilon}_2 \gamma^\mu \epsilon_1) \partial_\mu \begin{pmatrix} \phi \\ \phi^\dagger \end{pmatrix} \mp m(\bar{\epsilon}_2 \epsilon_1) \begin{pmatrix} \phi \\ \phi^\dagger \end{pmatrix} \\ [\delta(\epsilon_1), \delta(\bar{\epsilon}_2)]\psi &= (\bar{\epsilon}_2 \gamma^\mu \epsilon_1) \partial_\mu \psi - m(\bar{\epsilon}_2 \epsilon_1)\psi + \frac{1}{2}(\bar{\epsilon}_2 \epsilon_1)F - \frac{1}{2}(\bar{\epsilon}_2 \gamma^\mu \epsilon_1) \gamma_\mu F, \end{aligned} \quad (3.1)$$

where F is the complete field equation² for ψ ,

$$F = \not{\partial}\psi + m\psi \left(1 - \frac{2\phi^\dagger\phi}{\rho} \right) - \frac{2}{\rho} \gamma^\mu \psi (\phi^\dagger \partial_\mu \phi) - \frac{2}{\rho^2} (\bar{\psi}\psi)\psi. \quad (3.2)$$

(The susy commutator for $\bar{\psi}$ is easily derived by using $\delta\bar{\psi} = \delta\psi^\dagger i\gamma^0$.)

The above algebra has the expected form of

$$\{Q, \bar{Q}\} = \gamma^\mu P_\mu + iZ \quad (3.3)$$

where P_μ is the antihermitian translation generated represented by ∂_μ in (3.1) and Z is the anti-hermitian central charge proportional to the unit matrix which takes on the same value on both ϕ and ψ , because those are in the same multiplet (and opposite value on the complex conjugate multiplet with ϕ^\dagger and $\bar{\psi}$).

The susy currents can be derived from the Noether method, by letting the rigid ϵ become local. One finds

$$j^\mu = \frac{r}{\rho^2} \left[\gamma^\rho (\partial_\rho \phi^\dagger) + m\phi^\dagger \right] \gamma^\mu \psi, \quad \bar{j}^\mu = \frac{r}{\rho^2} \bar{\psi} \gamma^\mu \left[\gamma^\rho (\partial_\rho \phi) - m\phi \right]. \quad (3.4)$$

²Note that as in any nonlinear theory, the fermionic terms in the action do not vanish on-shell; rather on-shell a term $(\bar{\psi}\psi)(\bar{\psi}\psi)$ remains.

One may check that $\delta_{\bar{\epsilon}}\phi = [-i\bar{\epsilon}Q, \phi]$, $\delta_{\bar{\epsilon}}\bar{\psi} = [-i\bar{\epsilon}Q, \bar{\psi}]$ and $\delta_{\epsilon}\bar{\psi} = [-i\bar{Q}\epsilon, \bar{\psi}]$ with $Q = \int j^0 dx dy$ reproduce the transformation rules with canonical conjugate momenta

$$p(\phi) = \frac{r}{\rho^2}\dot{\phi}^\dagger + \frac{2r}{\rho^3}(\bar{\psi}\gamma^0\psi)\phi^\dagger, \quad p(\phi^\dagger) = \frac{r}{\rho^2}\dot{\phi}, \quad p(\psi) = \frac{r}{\rho^2}\bar{\psi}\gamma^0 \quad (3.5)$$

and $\{p(\psi)(t, \mathbf{x}), \psi(t, \mathbf{y})\} = -i\delta^2(\mathbf{x} - \mathbf{y})$. (No Dirac brackets are necessary if one uses (2.1) and replaces $\bar{\psi}$ by $p(\psi)$ as indicated, but note that (3.5) implies that $p(\phi^\dagger)$ is not equal to $(p(\phi))^\dagger$ if one uses naive hermitian conjugation.)

4. Classical CP¹ kink and domain line

The classical kink (domain wall) solution interpolating between the two minima $\phi = 0$ and $\phi = \infty$ of the potential (1.4) for the bosonic fields is most easily found by completing squares in the bosonic part of the classical Hamiltonian density. Assuming dependence of ϕ on only the x coordinate, we have

$$\mathcal{H} = \frac{r}{\rho^2}(\partial_x\phi^\dagger - m\phi^\dagger)(\partial_x\phi - m\phi) + \partial_x \left(\frac{-rm}{\rho} \right). \quad (4.1)$$

So the classical kink solution and its mass are

$$\phi_K = e^{m(x-x_0)+i\alpha}, \quad M_{\text{cl}} = rm. \quad (4.2)$$

There are two real moduli, x_0 and α , and correspondingly two real (one complex) zero modes, see (6.12).

The classical kink solution preserves one half of susy: from (2.5) with $\delta\psi = 0$ and $\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ we see that the remaining susy is given by $\epsilon = \begin{pmatrix} 0 \\ \epsilon^- \end{pmatrix}$. The broken susy with $\epsilon = \begin{pmatrix} \epsilon^+ \\ 0 \end{pmatrix}$ produces the fermionic zero mode

$$\psi \sim \phi_K \begin{pmatrix} \epsilon^+ \\ 0 \end{pmatrix}. \quad (4.3)$$

Since the generators of the preserved susy are $\bar{Q}\epsilon = -i(Q^+)^\dagger\epsilon^-$ and $\bar{\epsilon}Q = i(\epsilon^-)^\dagger Q^+$, we see that Q^+ and $(Q^+)^\dagger$ preserve the solitonic ground state $|sol\rangle$. BPS saturation at the quantum level thus requires

$$\langle sol | \{Q^+, (Q^+)^\dagger\} | sol \rangle = 0. \quad (4.4)$$

This implies that $\int (T^0_0 + T^0_2) dx dy$ should vanish. In the classical 2-dimensional model, T^0_2 is a regularized central charge density, and ζ^0 a second one. To evaluate them at the quantum level, we need to obtain the currents T^μ_ν and ζ^μ .

5. Energy momentum tensor and central charge currents

The variation $\delta(\bar{\epsilon})j^\mu$ vanishes, as one easily checks, but for $\delta(\epsilon)j^\mu$ we find, after tedious but straightforward algebra, using Fierz rearrangements but never discarding terms that are total derivatives, the following results

$$\delta(\epsilon)j^\mu = T^\mu_\nu \gamma^\nu \epsilon + \zeta^\mu \epsilon \quad (\mu, \nu = 0, 1, 2) \quad (5.1)$$

where

$$\begin{aligned}
T^\mu{}_\nu = \frac{r}{\rho^2} & \left[\partial^\mu \phi^\dagger \partial_\nu \phi + \partial_\nu \phi^\dagger \partial^\mu \phi - \delta_\nu^\mu (\partial^\lambda \phi^\dagger \partial_\lambda \phi + m^2 \phi^\dagger \phi) - \frac{1}{2} (\partial^\mu \bar{\psi}) \gamma_\nu \psi - \frac{1}{2} (\partial_\nu \bar{\psi}) \gamma^\mu \psi \right. \\
& + \frac{1}{\rho} (\partial^\mu \phi^\dagger) \phi \bar{\psi} \gamma_\nu \psi + \frac{1}{\rho} (\partial_\nu \phi^\dagger) \phi \bar{\psi} \gamma^\mu \psi - \delta_\nu^\mu \frac{1}{\rho^2} (\bar{\psi} \psi) (\bar{\psi} \psi) - \frac{1}{2} \delta_\nu^\mu \bar{F} \psi \\
& \left. + \epsilon^\mu{}_\nu{}^\lambda \left\{ m \partial_\lambda (\phi^\dagger \phi) - \frac{m}{2} \bar{\psi} \gamma_\lambda \psi \left(1 - \frac{2\phi^\dagger \phi}{\rho} \right) + \frac{1}{2} (\partial_\lambda \bar{\psi}) \psi - \frac{(\partial_\lambda \phi^\dagger) \phi}{\rho} \bar{\psi} \psi \right\} \right]. \quad (5.2)
\end{aligned}$$

Here \bar{F} is the complete field equation of $\bar{\psi}$,

$$\bar{F} = -\partial_\mu \bar{\psi} \gamma^\mu + m \bar{\psi} \left(1 - \frac{2\phi^\dagger \phi}{\rho} \right) + \frac{2}{\rho} \bar{\psi} (\not{\partial} \phi^\dagger) \phi - \frac{2}{\rho^2} (\bar{\psi} \psi) \bar{\psi}. \quad (5.3)$$

On-shell $T^\mu{}_\nu$ is not symmetric, nor should it be symmetric, for two reasons: it is not the gravitational stress tensor, and it may contain total derivatives which are antisymmetric in μ, ν . These total derivatives will contribute to the central charge. In order to obtain a $T^\mu{}_\nu$ which is symmetric up to total derivatives (and in which ψ and $\bar{\psi}$ appear on equal footing) one can proceed in two ways: either one adds $\delta(\bar{\epsilon}_1)(\bar{j}^\mu \epsilon_2)$ to $\delta(\epsilon_2)\bar{\epsilon}_1 j^\mu$ (which both come from $[\bar{\epsilon}_1 Q, \bar{Q} \epsilon_2]$) and divides by 2, or one partially integrates various terms in $T^\mu{}_\nu$, keeping track of total derivatives. The result is the same and reads

$$\begin{aligned}
T^\mu{}_\nu = \frac{r}{\rho^2} & \left[\partial^\mu \phi^\dagger \partial_\nu \phi + \partial_\nu \phi^\dagger \partial^\mu \phi - \delta_\nu^\mu (\partial^\lambda \phi^\dagger \partial_\lambda \phi + m^2 \phi^\dagger \phi) + \frac{1}{4} (\bar{\psi} \gamma^\mu \overleftrightarrow{\partial}_\nu \psi) + \frac{1}{4} (\bar{\psi} \gamma_\nu \overleftrightarrow{\partial}^\mu \psi) \right. \\
& - \frac{1}{2\rho} (\phi^\dagger \overleftrightarrow{\partial}^\mu \phi) \bar{\psi} \gamma_\nu \psi - \frac{1}{2\rho} (\phi^\dagger \overleftrightarrow{\partial}_\nu \phi) \bar{\psi} \gamma^\mu \psi - \delta_\nu^\mu \frac{1}{\rho^2} (\bar{\psi} \psi) (\bar{\psi} \psi) - \frac{1}{4} \delta_\nu^\mu (\bar{F} \psi + \bar{\psi} F) \left. \right] \\
& + r \epsilon^\mu{}_\nu{}^\lambda \partial_\lambda \left\{ -\frac{m}{\rho} + \frac{1}{4\rho^2} \bar{\psi} \psi \right\}. \quad (5.4)
\end{aligned}$$

The first two lines now correspond to the gravitational stress tensor, where all terms with δ_ν^μ can be written as $\delta_\nu^\mu \mathcal{L}$ with \mathcal{L} from (2.3), and the last term, which is a total derivative, is the only one antisymmetric in μ, ν . Note that although the various ways of writing the action, eqs. (2.1)–(2.4), differ by total derivatives, there is no ambiguity in the total derivatives in this $T^\mu{}_\nu$, because it is by definition due to the susy variation of the susy current j^μ , and the latter is unambiguous.³

The central charge current ζ^μ is found to be given by

$$\begin{aligned}
\zeta^\mu = \epsilon^{\mu\nu\lambda} & \frac{\partial_\nu \phi^\dagger \partial_\lambda \phi}{\rho^2} + \frac{m}{\rho^2} (\phi^\dagger \overleftrightarrow{\partial}^\mu \phi) - \frac{m}{2\rho^2} (\bar{\psi} \gamma^\mu \psi) \left(1 - \frac{2\phi^\dagger \phi}{\rho} \right) \\
& + \frac{1}{\rho^3} (\partial_\lambda \phi^\dagger) \phi \bar{\psi} \gamma^\lambda \gamma^\mu \psi - \frac{1}{2\rho^2} (\partial_\lambda \bar{\psi}) \gamma^\lambda \gamma^\mu \psi. \quad (5.5)
\end{aligned}$$

Again we can either partially integrate half of the last term, or subtract $\tilde{\zeta}^\mu$ (and divide by 2), where $\delta(\bar{\epsilon})\bar{j}^\mu = \tilde{T}^\mu{}_\nu (-\bar{\epsilon} \gamma^\nu) + \tilde{\zeta}^\mu \bar{\epsilon}$. The result is the same on-shell and reads

$$\zeta^\mu = \epsilon^{\mu\nu\lambda} \frac{\partial_\nu \phi^\dagger \partial_\lambda \phi}{\rho^2} + \frac{m}{\rho^2} \left[(\phi^\dagger \overleftrightarrow{\partial}^\mu \phi) - \bar{\psi} \gamma^\mu \psi \left(1 - \frac{2\phi^\dagger \phi}{\rho} \right) \right] + \frac{1}{2\rho^2} \bar{F} \gamma^\mu \psi, \quad (5.6)$$

where we used that $(\bar{\psi} \psi) (\bar{\psi} \gamma^\mu \psi) = 0$.

³We exclude topological terms in the susy current because they would lead to modifications of the susy transformations at the boundary.

6. Quantization

For the evaluation of one-loop quantum corrections we need to obtain the fluctuation equations in the CP^1 kink background ϕ_K .

The fermionic fluctuations satisfy the field equation (3.2), and to linear order in ψ with $\phi = \phi_K$ one has

$$\not{\partial}\psi + m\psi \left(1 - \frac{2\phi^\dagger\phi}{\rho}\right) - \frac{2}{\rho}\gamma^\mu\psi(\phi_K^\dagger\partial_\mu\phi_K) = 0. \quad (6.1)$$

Using the explicit form of the kink solution (4.2), with $x_0 = 0$ and $\alpha = 0$ for simplicity, and our representation of the γ matrices as given in (2.2) we obtain

$$\begin{pmatrix} \tilde{L} & -\partial_0 + \partial_y \\ \partial_0 + \partial_y & L \end{pmatrix} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = 0, \quad \begin{aligned} \tilde{L} &= -\partial_x + m, \\ L &= \partial_x + m - 4me^{2mx}/(1 + e^{2mx}). \end{aligned} \quad (6.2)$$

With respect to an inner product defined by $(\lambda, \chi) = \int \frac{1}{\rho^2}\lambda^*\chi dx$, the operator \tilde{L} is the adjoint of L , $(\lambda, L\chi) = (\tilde{L}\lambda, \chi)$ up to surface terms. Iterating (6.2) yields

$$(L\tilde{L} - \partial_y^2 + \partial_0^2)\psi^+ = 0, \quad (6.3)$$

$$(\tilde{L}L - \partial_y^2 + \partial_0^2)\psi^- = 0. \quad (6.4)$$

The operators $L\tilde{L}$ and $\tilde{L}L$ are selfadjoint without surface terms, so they yield a complete set of eigenfunctions. Let $\varphi_k(x)$ be a solution of

$$L\tilde{L}\varphi_k = \omega_k^2\varphi_k \quad \text{with } \omega_k^2 = k^2 + m^2, \quad (6.5)$$

and let

$$s_k = \frac{1}{\omega_k}\tilde{L}\varphi_k. \quad (6.6)$$

Then in second quantization

$$\begin{aligned} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} &= \frac{1}{\sqrt{r}} \int \frac{dk d^\epsilon\ell}{(2\pi)^{(1+\epsilon)/2}} \frac{1}{\sqrt{2\omega}} \left[\alpha_{k\ell} \begin{pmatrix} \sqrt{\omega + \ell}\varphi_k(x) \\ \sqrt{\omega - \ell}is_k(x) \end{pmatrix} e^{i\ell y - i\omega t} \right. \\ &\quad \left. + \beta_{k\ell}^\dagger \begin{pmatrix} \sqrt{\omega + \ell}\varphi_k^*(x) \\ -\sqrt{\omega - \ell}is_k^*(x) \end{pmatrix} e^{-i\ell y + i\omega t} \right] \\ &\quad + \frac{1}{\sqrt{r}} \int \frac{d^\epsilon\ell}{(2\pi)^{\epsilon/2}} \gamma_\ell \begin{pmatrix} \varphi_0(x) \\ 0 \end{pmatrix} e^{i\ell(y-t)}, \end{aligned} \quad (6.7)$$

where $\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$ satisfies (6.2), and $\omega^2 \equiv k^2 + \ell^2 + m^2$. Here ℓ is the momentum component along the domain wall, and we have already indicated that dimensional regularization by dimensional reduction will eventually be performed by sending ϵ from 1 to 0. The last term is due to the fermionic zero mode, which in dimensions larger than 2 turns into a continuum of massless modes localized along the domain line and with definite chirality with respect to the latter. The correct normalization of this term can be obtained by taking the formal limit $\omega_k \rightarrow 0$ in the nonzero mode terms and combining the terms with $\ell > 0$

and $\ell < 0$ into one term with $-\infty < \ell < \infty$, setting $\{\gamma_\ell, \gamma_{\ell'}^\dagger\} = \delta(\ell - \ell')$. Note that γ_ℓ (γ_ℓ^\dagger) have the meaning of annihilation (creation) operators only for $\ell > 0$ and that for $\ell < 0$ this is to be reversed. As (6.7) shows, the positive frequency modes have momentum in positive y -direction only, so that there is a breaking of parity invariance with respect to the regulator dimension. The opposite breaking would have taken place with the choice $\gamma^2 = -\sigma^1$, which gives a nonequivalent second representation of the Clifford algebra in 3 dimensions.

The bosonic fluctuations η are obtained from $\phi = \phi_K + \eta$, and after some work one finds for their linearized field equations the same result as for ψ^+ ,

$$(L\tilde{L} - \partial_y^2 + \partial_0^2)\eta = 0. \tag{6.8}$$

To solve this equation we first look at its behaviour at large $|x|$, where $L\tilde{L} \rightarrow -\partial_x^2 + 4m\partial_x - 3m^2$ as $x \rightarrow +\infty$ and $L\tilde{L} \rightarrow -\partial_x^2 + m^2$ as $x \rightarrow -\infty$. We set then

$$\eta(x) = (1 + e^{2mx})g(x) \tag{6.9}$$

and find for $g(x)$ the differential equation

$$\left[-\partial_x^2 + m^2 - \frac{2m^2}{\cosh^2(mx)}\right]g = \omega_k^2 g. \tag{6.10}$$

This is the $l = 1$ case of the sequence of operators

$$\mathcal{O}_l = A_l^\dagger A_l = -\partial_z^2 + l^2 - \frac{l(l+1)}{\cosh^2 z} \tag{6.11}$$

with $A_l = \partial_z + l \tanh z$ and $A_l^\dagger = -\partial_z + l \tanh z$, where $z = mx$. For $l = 1$, this system, which also appears in the 2-dimensional sine-Gordon model,⁴ contains one zero mode, no bound state, and a continuum of solutions, given respectively by

$$g_0(x) = \sqrt{\frac{m}{2}} \frac{1}{\cosh(mx)}, \tag{6.12}$$

$$g_k(x) = \frac{1}{\sqrt{2\pi}} \frac{-ik + m \tanh(mx)}{\omega_k} e^{ikx}. \tag{6.13}$$

Note that g_0 corresponds to $\varphi_0(x) = \rho_K(x)g_0(x) = \sqrt{2m}e^{mx}$ which is indeed proportional to the function arising from differentiating ϕ_K in (4.2) with respect to either of the moduli x_0 or α .

Then in second quantization

$$\begin{aligned} \eta(t, x, y) = & \frac{1}{\sqrt{r}} \int \frac{dk d^\epsilon \ell}{(2\pi)^{(1+\epsilon)/2}} \frac{1}{\sqrt{2\omega}} \left[a_{kl} \varphi_k(x) e^{i\ell y - i\omega t} + b_{kl}^\dagger \varphi_k^*(x) e^{-i\ell y + i\omega t} \right] \\ & + \frac{1}{\sqrt{r}} \int \frac{d^\epsilon \ell}{(2\pi)^{\epsilon/2}} \frac{1}{\sqrt{2|\ell|}} \left[c_\ell \varphi_0(x) e^{i\ell y - i|\ell|t} + d_\ell^\dagger \varphi_0(x) e^{-i\ell y + i|\ell|t} \right], \end{aligned} \tag{6.14}$$

⁴The sine-Gordon model also appears in the dual formulation of the CP¹ model [34–36]

with $\omega^2 = \omega_k^2 + \ell^2 = k^2 + \ell^2 + m^2$. Given the normalization of $g_k(x)$ to plane waves at infinity, we have the following orthonormality relations

$$\int \frac{dx}{\rho^2(x)} \varphi_0^2(x) = 1, \quad \int \frac{dx}{\rho^2(x)} \varphi_k^*(x) \varphi_{k'}(x) = \delta(k - k'), \quad \int \frac{dx}{\rho^2(x)} \varphi_0(x) \varphi_k(x) = 0. \quad (6.15)$$

We shall also need the difference of the spectral densities associated with the continuum solutions φ_k and s_k , which is defined by

$$\Delta\sigma(k) = \int \frac{dx}{\rho^2(x)} (|\varphi_k(x)|^2 - |s_k(x)|^2). \quad (6.16)$$

Using $s_k = \frac{1}{\omega_k} \tilde{L} \varphi_k$ and partially integrating, only a surface term is left, and we find

$$\Delta\sigma(k) = \frac{\varphi_k^* \tilde{L} \varphi}{\omega_k^2 \rho^2(x)} \Big|_{x=-\infty}^{x=\infty} = \frac{-2m}{\omega_k^2} = \frac{-2m}{k^2 + m^2}. \quad (6.17)$$

This result agrees with the analysis of ref. [5], where a nonlinear transformation of the fluctuating fields was employed that simplified the fluctuation equations, but which corresponds to a reparametrization of the fields that cannot be used in perturbation theory about the topologically trivial vacuum, where the renormalization of the model is to be fixed (one of the real fields has no kinetic term in the vacuum). Our approach thus has the advantage of not having to combine results from calculations using different parametrizations of the target space, but a posteriori we find that no mistake would have been made by doing so.

7. The mass of the CP¹ kink

The classical kink mass $M_{cl} = rm$ gets quantum corrections from the zero point energies of the fluctuating fields and from renormalization,

$$M^{(1)} = \int dx \langle T_{00}^{(1)} \rangle + \frac{\Delta r}{r} M_{cl}, \quad (7.1)$$

where the subscript (1) refers to one-loop order contributions and where we have anticipated that only r and not m gets renormalized in our model, which is in fact true to all orders in perturbation theory [37].

The one-loop renormalization $r_0 = r + \Delta r$ of the coupling constant $r \equiv 2/g^2$ can be obtained from the scalar self energy corrections (or equivalently from the fermionic ones) in the trivial vacuum. Imposing the renormalization condition that they vanish fixes Δr ,

$$\text{---}\overset{\curvearrowright}{\text{---}}\text{---} + \text{---}\overset{\curvearrowleft}{\text{---}}\text{---} + \text{---}\overset{\Delta r}{\times}\text{---} = 0. \quad (7.2)$$

By straightforward calculation we find

$$\text{---}\overset{\curvearrowright}{\text{---}}\text{---} + \text{---}\overset{\curvearrowleft}{\text{---}}\text{---} = 2 \int \frac{d^{2+\epsilon}k}{(2\pi)^{2+\epsilon}} \frac{p^2 + m^2 - (k^2 + m^2)}{k^2 + m^2 - i\epsilon}. \quad (7.3)$$

The integral with $-(k^2 + m^2)$ in the numerator vanishes in dimensional regularization, whereas the terms with $p^2 + m^2$ can be canceled by a counterterm Δr , leaving m unrenormalized. This leads to

$$\Delta r = \int \frac{dk d^\epsilon \ell}{(2\pi)^{1+\epsilon}} \frac{1}{\omega}, \quad \omega = \sqrt{k^2 + \ell^2 + m^2}, \quad (7.4)$$

where the sign of this result corresponds to the well-known asymptotic freedom of this model.

The bulk contributions to the mass are given by

$$\langle T_{00}^{(1)} \rangle = \left\langle \frac{r}{\rho^2} \left(\partial_0 \phi^\dagger \partial_0 \phi + \partial_k \phi^\dagger \partial_k \phi + m^2 \phi^\dagger \phi - \frac{1}{2} \bar{\psi} \gamma^0 \overleftrightarrow{\partial}_0 \psi \right) \right\rangle \quad (7.5)$$

where we dropped the terms with the fermionic field equations. Rewriting the bosonic terms in this expression as $(2r\rho^{-2}\partial_0\phi^\dagger\partial_0\phi - \mathcal{L})$ and using that for any action $\langle \mathcal{L}_{ferm.}^{(2)} \rangle = 0$ but $\mathcal{L}_{bos}^{(2)} = 0$ only up to boundary terms, we can recast $\langle T_{00}^{(1)} \rangle$ as follows

$$\langle T_{00}^{(1)} \rangle = \frac{r}{\rho^2} \left\langle 2\partial_0 \eta^\dagger \partial_0 \eta - \bar{\psi} \gamma^0 \partial_0 \psi \right\rangle + \text{total derivatives} \quad (7.6)$$

The total derivatives are given by

$$r\partial_x \left[\frac{m\phi_K^2}{\rho_K^3} (\eta + \eta^\dagger)^2 \right] - r\partial_\mu \left[\frac{\eta\partial^\mu\eta^\dagger}{\rho_K^2} \right], \quad (7.7)$$

but they do not contribute to the energy. (The propagator $\langle \eta\eta^\dagger \rangle$ is proportional to ρ_K^2 , and the derivatives of ρ_K in the second term cancel the first term. One is left with a ρ -independent term with a derivative on the distorted plane wave, and this term is the same at plus and minus infinity.)

Substituting the mode expansion of η and ψ yields

$$\begin{aligned} M_{\text{bulk}}^{(1)} &= \int dx \langle T_{00}^{(1)} \rangle = \int \frac{dx}{\rho^2} \int \frac{dk d^\epsilon \ell}{(2\pi)^{1+\epsilon} 2\omega} [2\omega^2 |\varphi_k|^2 - \omega \{(\omega + \ell) |\varphi_k|^2 + (\omega - \ell) |s_k|^2\}] \\ &= \int \frac{dx}{\rho^2} \int \frac{dk d^\epsilon \ell}{(2\pi)^{1+\epsilon}} \frac{\omega}{2} (|\varphi_k(x)|^2 - |s_k(x)|^2) = - \int \frac{dk d^\epsilon \ell}{(2\pi)^{1+\epsilon}} \frac{m\omega}{\omega_k^2}, \end{aligned} \quad (7.8)$$

where we used the expression for the difference of spectral densities obtained in eq. (6.17). We see here clearly the sums over zero-point energies ($\sum \hbar\omega$ for complex scalars, $-\sum \hbar\omega$ for complex fermions) and that despite of supersymmetry and isospectrality there is a net contribution due to a difference of the spectral density of the continuum modes. This contribution is in fact ultraviolet divergent and becomes finite upon combining it with the counterterm $\Delta r m$. Using the integral representation of Δr of eq. (7.4) the total mass correction is given by

$$\begin{aligned} M^{(1)} &= m \int \frac{dk d^\epsilon \ell}{(2\pi)^{1+\epsilon}} \left(\frac{-m\omega}{\omega_k^2} + \frac{m}{\omega} \right) = -m \int \frac{dk d^\epsilon \ell}{(2\pi)^{1+\epsilon}} \frac{\ell^2}{\omega\omega_k^2} \\ &= -\frac{4}{1+\epsilon} \frac{\Gamma(1-\epsilon/2)}{(4\pi)^{1-\epsilon/2}} m^{1+\epsilon/2} = -\frac{m}{\pi} + O(\epsilon), \end{aligned} \quad (7.9)$$

which is finite for all $\epsilon < 2$. For $\epsilon = 0$ one obtains the nonvanishing correction $M^{(1)} = -m/\pi$ for the mass of the susy CP^1 kink; for $\epsilon = 1$ the result corresponds to the mass per unit length of the domain line and then reads $-m^2/(4\pi)$. Both results are precisely twice the universal⁵ amount one finds for minimally supersymmetric 1+1-dimensional kinks and 2+1-dimensional domain lines, respectively, provided the latter are renormalized in a minimal scheme [26]. By contrast, ordinary $\mathcal{N} = 2$ susy kinks in Landau-Ginzburg type models lead to complete cancellations of the quantum corrections [22] instead of the doubling we found here for the $\mathcal{N} = 2$ nonlinear sigma model with twisted mass term.

Next we shall consider the quantum corrections to the central charges, which have to involve the same finite correction in order that BPS saturation holds. This will moreover show that these finite corrections are associated with an anomaly.

8. The central charges

The central charge responsible for the saturation of the BPS bound is associated with T^0_2 of the 3-dimensional model, as follows from (4.4). Its evaluation now involves bulk contributions, boundary terms, and a renormalization term,

$$T^0_2 = \frac{r}{\rho^2} \left[-\partial_0 \phi^\dagger \partial_2 \phi - \partial_2 \phi^\dagger \partial_0 \phi + \frac{1}{4} \bar{\psi} \gamma^0 \overleftrightarrow{\partial}_2 \psi - \frac{1}{4} \bar{\psi} \gamma^2 \overleftrightarrow{\partial}_0 \psi \right] + r \partial_x \left(\frac{m}{\rho} - \frac{\bar{\psi} \psi}{4\rho^2} \right) + \Delta r \partial_x \frac{m}{\rho}. \tag{8.1}$$

As is usual for central charge corrections in susy models [20], loop corrections from the bosonic surface terms cancel the renormalization term exactly,

$$r \left\langle \frac{m}{\rho} \right\rangle \Big|_{-\infty}^{\infty} = r \frac{m}{\rho^3} 2\phi^\dagger \langle \eta \eta^\dagger \rangle \phi \Big|_{-\infty}^{\infty} = \int \frac{dk d^\ell \ell}{(2\pi)^{1+\epsilon}} \frac{m}{\omega} = m \Delta r = -\Delta r \frac{m}{\rho} \Big|_{-\infty}^{\infty}. \tag{8.2}$$

Quite unusually, the fermionic surface term does contribute and is even divergent,

$$-\frac{r}{4\rho^2} \langle \bar{\psi} \psi \rangle \Big|_{-\infty}^{\infty} = \frac{1}{\rho^2} \int \frac{dk d^\ell \ell}{(2\pi)^{1+\epsilon}} \frac{\omega_k}{8\omega} (\varphi_k s_k^* + s_k \varphi_k^*) \Big|_{-\infty}^{\infty} = \frac{1}{\rho^2} \int \frac{dk d^\ell \ell}{(2\pi)^{1+\epsilon}} \frac{1}{8\omega} (-2\rho \partial_x \rho + 2m\rho^2) \Big|_{-\infty}^{\infty} = -\frac{m}{2} \int \frac{dk d^\ell \ell}{(2\pi)^{1+\epsilon}} \frac{1}{\omega}. \tag{8.3}$$

The bosonic bulk terms vanish since they are odd in ℓ , but the fermionic bulk terms do contribute a nonvanishing momentum density along the domain line as follows,

$$\begin{aligned} & \frac{-i}{2} r \int \frac{dx}{\rho^2} \int \frac{dk d^\ell \ell}{(2\pi)^{1+\epsilon}} \left\langle (\psi^+)^\dagger (\partial_2 - \partial_0) \psi^+ + (\psi^-)^\dagger (\partial_2 + \partial_0) \psi^- \right\rangle \\ & = -\frac{1}{2} \int \frac{dx}{\rho^2} \int \frac{dk d^\ell \ell}{(2\pi)^{1+\epsilon} 2\omega} (\omega^2 + \ell^2) (|\varphi_k|^2 - |s_k|^2) = \int \frac{dk d^\ell \ell}{(2\pi)^{1+\epsilon}} \frac{(\omega^2 + \ell^2) m}{2\omega \omega_k^2}, \end{aligned} \tag{8.4}$$

⁵Because of supersymmetry the difference in the spectral densities which is responsible for the nonzero result is determined by the asymptotic values of the fermion mass and does not depend on other details of the potential [21, 22].

where once again (6.17) has been used. The total central charge Z_1 is finite and given by

$$Z_1^{(1)} = m \int \frac{dk d^\epsilon \ell}{(2\pi)^{1+\epsilon}} \frac{\omega^2 + \ell^2 - \omega_k^2}{2\omega\omega_k^2} = m \int \frac{dk d^\epsilon \ell}{(2\pi)^{1+\epsilon}} \frac{\ell^2}{\omega\omega_k^2}. \quad (8.5)$$

Comparing with (7.9), we see that BPS saturation holds, $M^{(1)} + Z_1^{(1)} = 0$.

The other central charge is $Z_2 = \int \zeta^0 dx$, where according to (5.6)

$$\zeta^0 = \epsilon^{0\nu\lambda} \frac{\partial_\nu \phi^\dagger \partial_\lambda \phi}{\rho^2} + \frac{m}{\rho^2} \left[(\phi^\dagger \overleftrightarrow{\partial}^0 \phi) - \bar{\psi} \gamma^0 \psi \left(1 - \frac{2\phi^\dagger \phi}{\rho} \right) \right]. \quad (8.6)$$

It generates the m -dependent terms in (3.1). Considering one-loop corrections, one finds that in momentum space the first term gives rise to an expression which is odd in ℓ and thus gives no contribution. The second term gives rise to

$$\frac{2m}{\rho^2} \langle \eta^\dagger \partial_0 \eta \rangle - \frac{4m}{\rho^3} \phi^\dagger \langle \partial_0 \eta \eta^\dagger \rangle \phi \quad (8.7)$$

and these terms vanish because they are independent of the extra momentum ℓ , leading to a scaleless integral which is zero in dimensional regularization. The contribution from the third term (8.6) is also ℓ -independent, because the ℓ in $(\omega + \ell)|\varphi_k|^2$ and $(\omega - \ell)|s_k|^2$ (produced by the mode expansion (6.7)) cancels by symmetric integration, after which the remaining ω cancels the energy denominator $\frac{1}{2\omega}$. Hence, the second central charge does not receive any one-loop corrections.

9. Discussion and conclusions

As mentioned in the Introduction, an exact result for the central charge of the quantum CP^1 kink in the nonlinear sigma model with a twisted mass term has been obtained by Dorey [5] in a generalization of results of Hanany and Hori [6], which for the kink configuration reads

$$\langle Z \rangle = \frac{1}{\pi} \sqrt{\tilde{m}^2 + 4\tilde{\Lambda}^2} + \frac{\tilde{m}}{2} \ln \frac{\tilde{m} - \sqrt{\tilde{m}^2 + 4\tilde{\Lambda}^2}}{\tilde{m} + \sqrt{\tilde{m}^2 + 4\tilde{\Lambda}^2}}, \quad (9.1)$$

where $\tilde{m} = m e^{i\beta}$ is the complex twisted mass parameter mentioned in the Introduction, and $\tilde{\Lambda}$ is the renormalization-group invariant scale of the model, which is real in the absence of a theta term. With the identification $r = 2g^{-2} = \frac{1}{2\pi} \ln(m^2/\tilde{\Lambda}^2)$, the weak-coupling limit of (9.1) corresponds to $m \gg \tilde{\Lambda}$, and expanding (9.1) in this limit yields

$$|\langle Z \rangle| = \left| \tilde{m} \frac{1}{2\pi} \ln \left(-\frac{\tilde{m}^2}{\tilde{\Lambda}^2} \right) - \frac{\tilde{m}}{\pi} \right|. \quad (9.2)$$

Identifying our (real) mass parameter m with $|\tilde{m}|$ and choosing $|\beta| = \pi/2$ such that the logarithm is real, (9.2) reduces to $|\langle Z \rangle| = rm - m/\pi$, in agreement with our real results for the one-loop correction of mass and central charge, (7.9) and (8.5).⁶

⁶A possible theta angle appears in the exact result (9.1) of ref. [5] as a phase of $\tilde{\Lambda}$ in such a way that the phase of \tilde{m} can be absorbed by a change of θ . However, using our scheme of dimensional regularization by embedding the kink in one higher dimension we have to restrict ourselves to $\theta = 0$.

The possible imaginary part in $\langle Z \rangle$ has to be identified with the second central charge, $Z_2 = \int dx \zeta^0$, considered above, which contains the Noether charge density for the global U(1) symmetry $\psi \rightarrow e^{i\lambda}\psi$, $\phi \rightarrow e^{i\lambda}\phi$ of (2.1). Besides the “purely magnetic” kink (4.2), this model also contains dyons, which are given by replacing the constant α by $\alpha(t) = \omega t$ in (4.2), where at the quantum level ω is quantized by a Bohr-Sommerfeld condition. In the above, we have considered a purely magnetic kink, but the exact result (9.1) shows that for general β (and also for general θ) one has dyonic states. In our calculation we have not obtained a contribution to Z_2 so that our result corresponds to a purely imaginary \tilde{m} in (9.1). Such a null result for the U(1) charge of the solitonic ground state does not contradict the fact that the latter should be defined as carrying fractional fermion number [38] because of the presence of fermionic zero modes. Indeed, the U(1) charge associated with the fermionic zero mode vanishes:

$$r \int \frac{dx}{\rho^2} \langle -\bar{\psi}\gamma^0\psi \left(1 - \frac{2\phi^\dagger\phi}{\rho}\right) \rangle = -2mr \int \frac{dx}{(1 + e^{2mx})^2} e^{2mx} \left(1 - \frac{2e^{2mx}}{1 + e^{2mx}}\right) = 0, \quad (9.3)$$

whereas the fermion number charge density is given by $\frac{r}{\rho^2}\bar{\psi}\gamma^0\psi$ (and in strictly two dimensions this gives a nonvanishing integral when the fermionic zero mode is inserted).

The final result that we have obtained for the one-loop correction to the mass of the kink, eq. (7.9), and correspondingly for the correction of one of the central charges, eq. (8.5), is given by $-m/\pi$. In the calculation of the previous section where we considered the central charges we have identified this contribution as arising from a net momentum density associated with fermionic modes along the domain line (whereas the classical contribution to the central charge is a pure surface term). Thus at the quantum level there is a breaking of parity in the extra regulator dimension which is induced by the kink background, similar to what occurs in the minimally susy kink [17].

Compared to previous calculations of quantum corrections to two-dimensional susy kinks we have noticed in particular two new features of the $\mathcal{N} = 2$ CP¹ model with twisted mass term: whereas in other $\mathcal{N} = 2$ susy kink models extended susy leads to a cancellation of the anomalous contributions [22, 26], in the $\mathcal{N} = 2$ CP¹ model they add up. Related to this is the fact that in the $\mathcal{N} = 2$ CP¹ model the complex fermion zero mode has definite chirality with respect to the domain line employed in our dimensional regularization scheme. Another noteworthy difference to other susy kinks is the appearance of fermionic surface terms in the one-loop corrections to the central charge, cf. eq. (8.3), which neither occurred in other susy kink models considered so far nor in the case of 4-dimensional (Coulomb phase) BPS monopoles, which with $\mathcal{N} = 2$ also receive anomalous contributions to their central charge [18].

To conclude, we have presented an explicit calculation of the one-loop corrections to both mass and central charge of the susy kink of the $\mathcal{N} = 2$ nonlinear sigma model with twisted mass and found agreement with the exact results obtained by Dorey in ref. [5]. The nontrivial corrections have been identified as being associated with an anomalous contribution to the central charge [14] that in our scheme appears as parity violation in the higher dimension used to imbed the susy kink as a domain line, which carries chiral domain wall fermions. This mechanism is completely parallel to the anomalous

contributions obtained in the minimally susy kink in 2 dimensions as well as the $\mathcal{N} = 2$ susy 't Hooft-Polyakov monopole [18], where the anomalous contribution to the central charge is required for consistency with the Seiberg-Witten solution. Indeed, as explained in ref. [11], holomorphicity relates the latter to the anomalous central charge of the nonabelian confined monopoles appearing in the Higgs phase of $\mathcal{N} = 2$ $SU(2) \times U(1)$ theory, whose effective low energy theory is given by the kinks of the two-dimensional $\mathcal{N} = 2$ CP^1 model with twisted mass.

Acknowledgments

We would like to thank Nick Dorey and Arkady Vainshtein for useful discussions. R.W. and P.v.N. have been supported by the Austrian Science Foundation FWF, project no. J2660 and NSF grant no. PHY-0354776, respectively.

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